ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

RESCIF
Réseau d'excellence des sciences de l'ingénieur de la Francophonie

Thermodynamique

Transformations de Legendre

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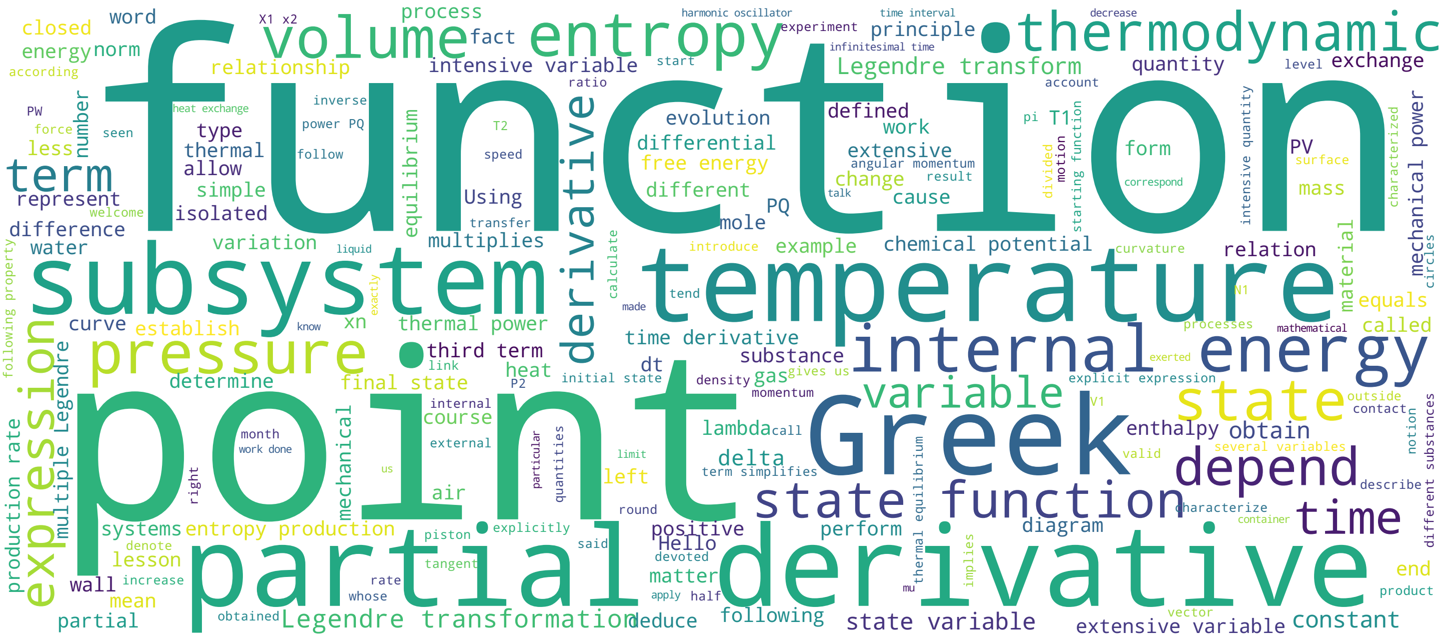
Hermann Ludwig Ferdinand von Helmoltz, 1821 - 1894



Video

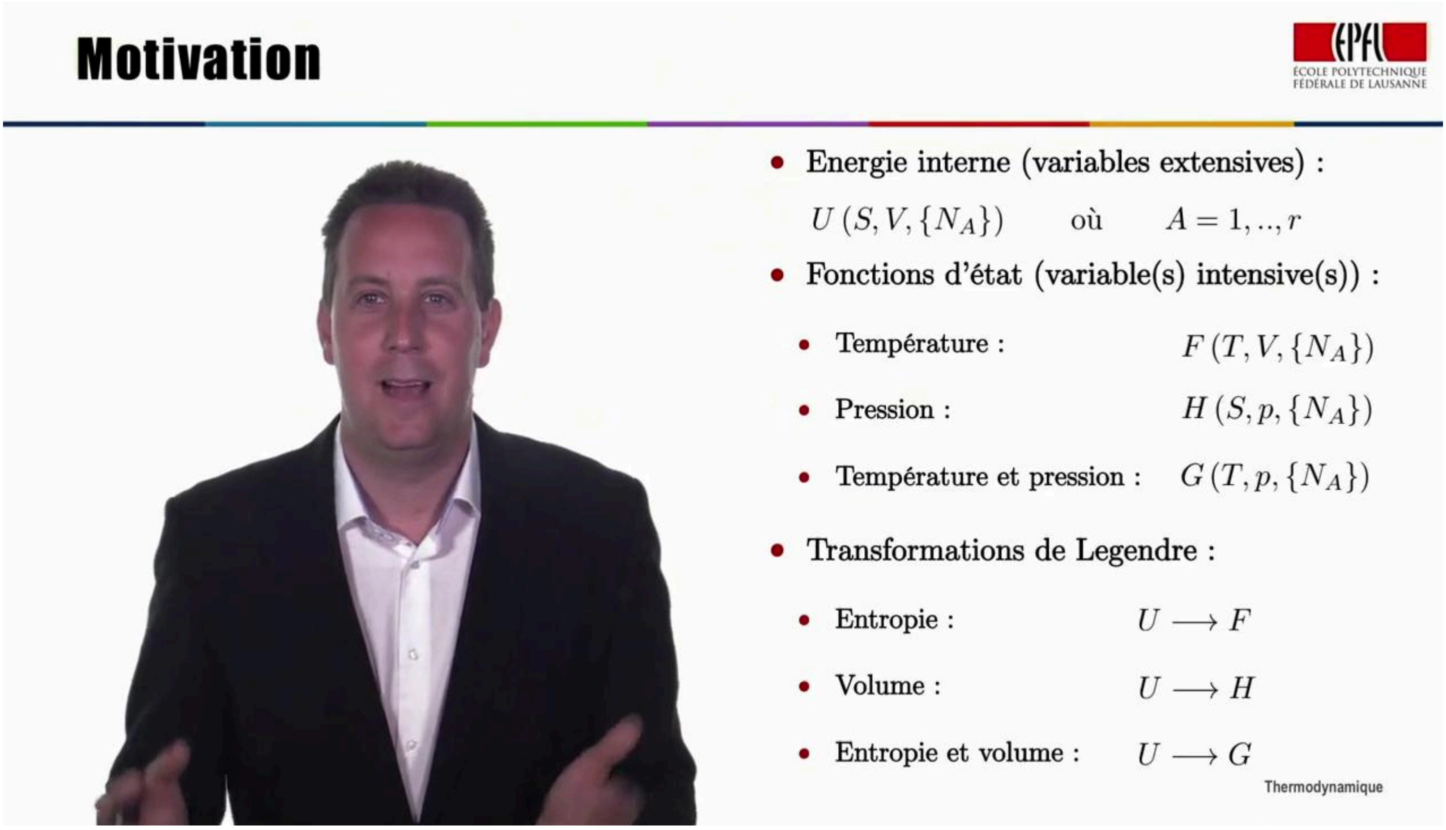
EPFL

Hello and welcome to this word of thermodynamics. This lesson is devoted to Legendre transformations. First of all, we will physically motivate these Legendre transformations which are mathematical. In a second step, we will establish the Legendre transform of a function of one variable. We will then in a third step extend this to a function of several variables. And then, in a fourth step, we will establish an explicit link between the Legendre transforms and their curves and finally, we will define a multiple Legendre transform.

Notes

Summary

0m 05s

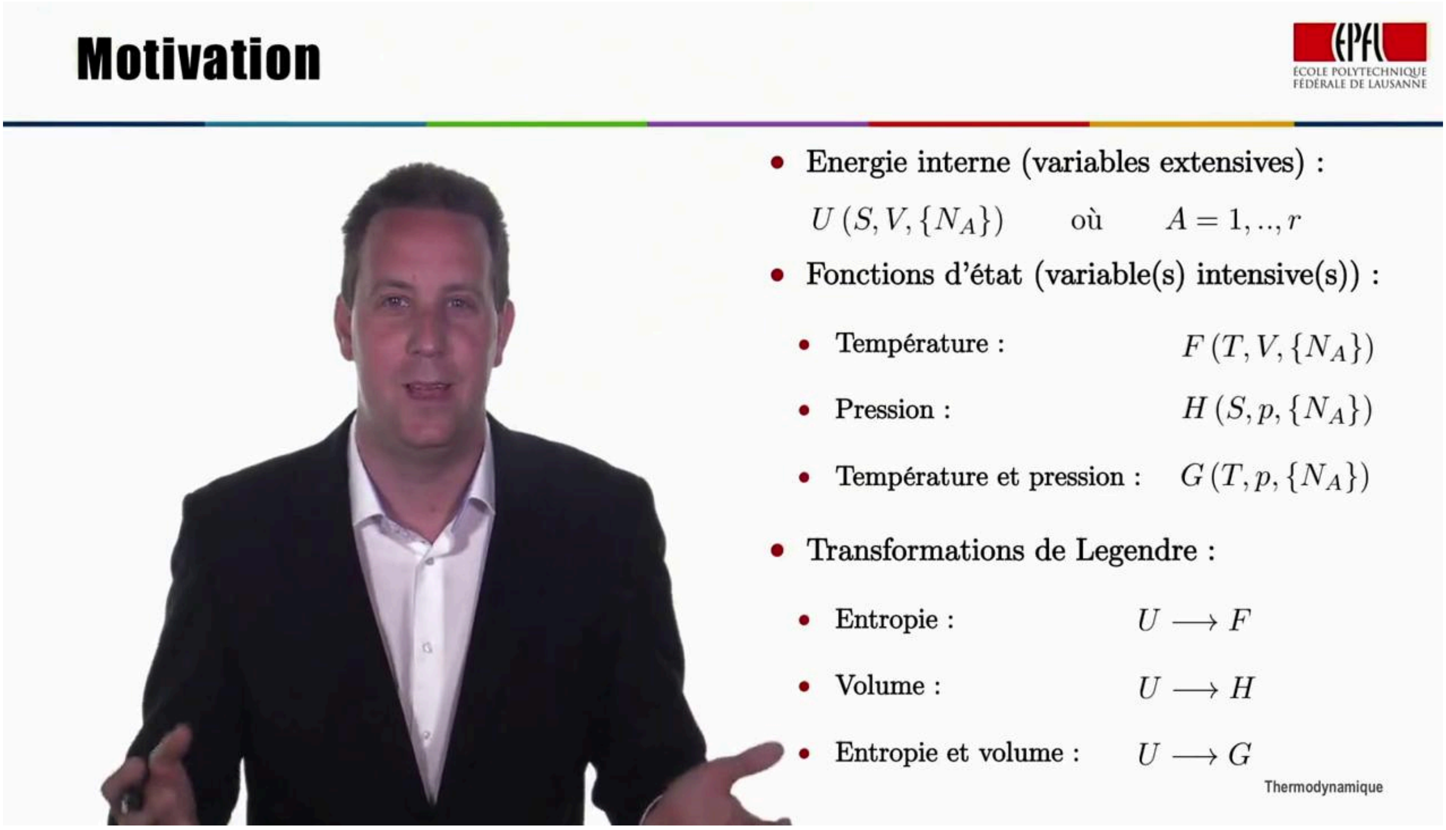


Internal energy plays a major role in thermodynamics. Internal energy is a state function of extensive variable. U is a function of the entropy S, the volume V and the set of numbers of moles of the different substances in the system. We would like to obtain state functions which have the same physical dimension as the internal energy. These are therefore other forms of energy that depend on one or more intensive variables. For example, we would like a state function that depends on temperature. Let's call this state function f. F will therefore be a function of T, V and the set DN. We would also like a state function which depends on the pressure calling this function. H h will then be a function of s, p and the set of norms. We would also like a which depend on temperature and pressure. Let us call this function G. G will thus be a function endowed with P and the set of N1. The functions F, H, LG. Are obtained by Legendre transformation of the internal energy. It is said to be the Legendre transforms. The Legendre transformation of the internal energy U with respect to the entropy gives the function f which depends on the temperature.

Notes

Summary

0m 46s



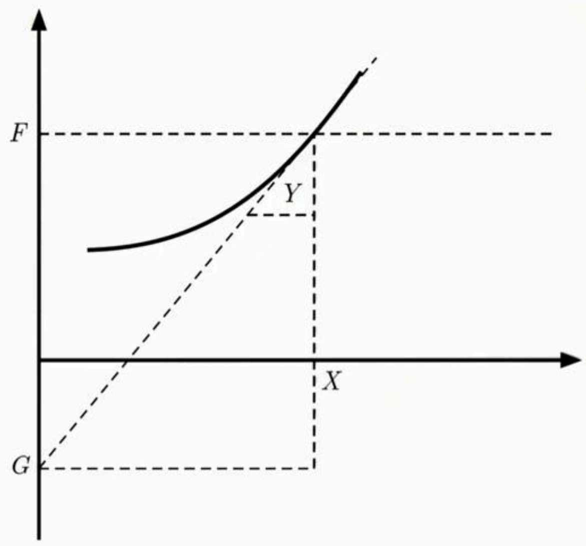
The Legendre transformation of the internal energy U in relation to the volumes gives the function H which depends on the pressure P. And the transformation of Legendre of the internal energy U in relation to the entropy and the volume gives the function G which depends on the temperature T and the pressure P.

Notes

Summary

2m 23s

Fonction d'une variable



- **Fonction d'état (variable extensive):**

  $F(X)$

- **Transformée de Legendre (variable intensive) :**

  $G(Y) \quad$ où $\quad F \longrightarrow G$

- **Propriété :**

  $$Y(X) = \frac{dF(X)}{dX}$$

- **Identité :**

  $$Y = \frac{F - G}{X - 0}$$

- **Transformée de Legendre $G(Y)$ de $F(X)$:**

  $$\boxed{G(Y) = F(X(Y)) - Y\,X(Y)}$$

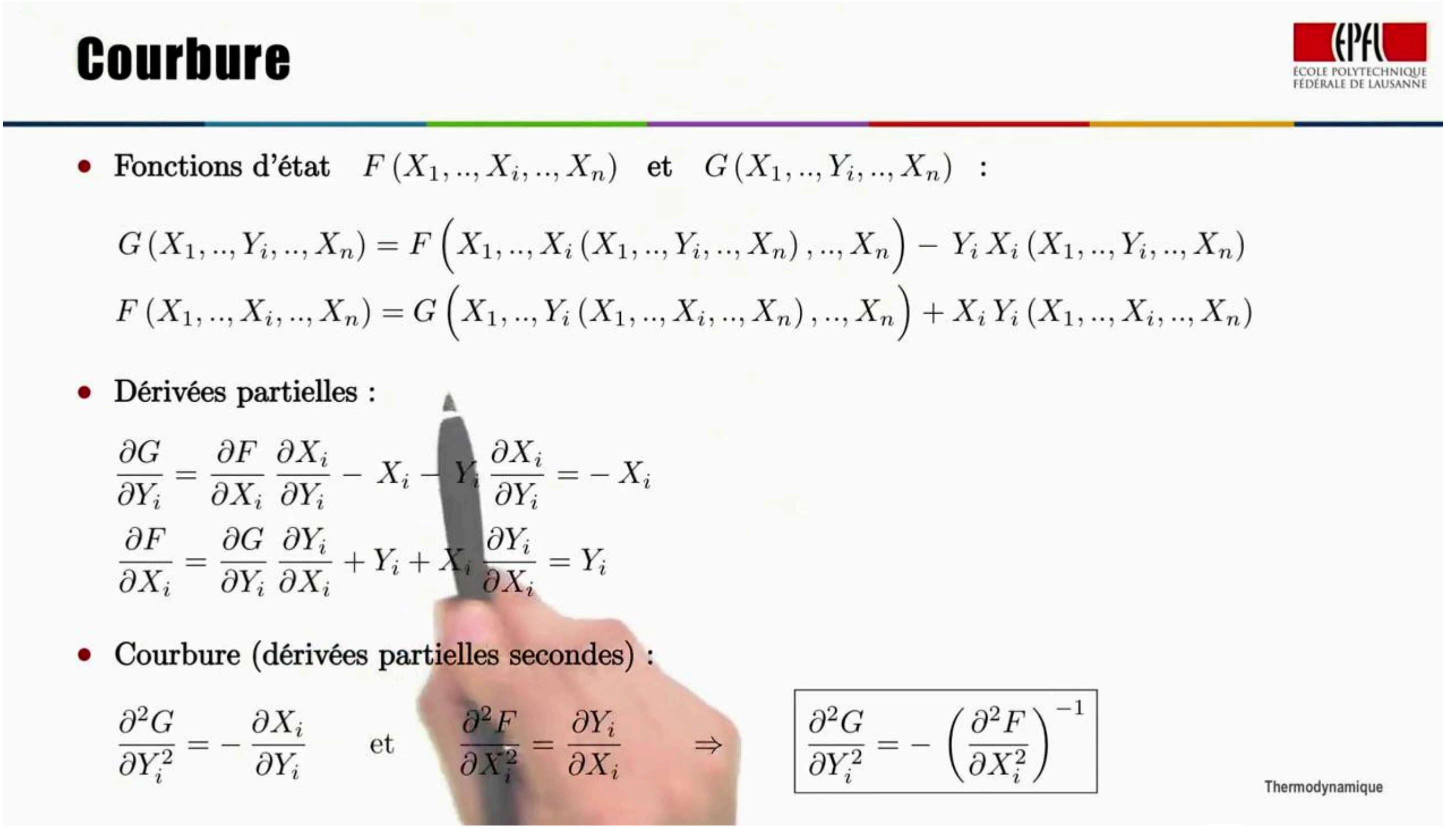
Thermodynamique

We will now deduce the expression of the Legendre transform of a function. Then we take a state function f which depends on a single variable x which is an extensive variable, and we want to obtain the expression of the Legendre transform of F which is the function G and which depends on an intensive variable which is I. I. And the quantity conjugate to X with respect to f. It therefore has the following property i of X and the derivative of f of x with respect to x. We will now represent this on a diagram. So we have the function f of x. This is the curve we see here. And we draw a tangent to this curve at a given point and this dashed tangent will intercept the vertical axis at point G, the slope of this tangent, this Greek i. Greek Y is defined as the ratio of the difference in level, i.e. f minus g since g is negative. Divided by the horizontal distance, i.e. x. Me zero. Using this identity, we can now obtain an explicit expression of the Legendre transform G of Greek I of the function f of x or x is a function of Greek i g from i greek seven, f from x from y greek times x y greek.

Notes

Summary

2m 49s



This result, we can now generalize it to a function of several variables. We therefore take a state function f. Which is a function of n extensive variables X1, x2, etc. until XN and we want to obtain the transform of the function f with respect to the variable x y. The new variable is the Greek i variable. It is an intensive variable. All other variables, all other x's are unchanged. There is therefore at least one extensive variable in the expression of G, which is an intensive quantity that is conjugate to x y. By definition, it therefore has the following property. I Greek i is equal to the partial derivative of f with respect to x y. We could also represent this on a graph and we would have exactly the same situation as for a function of one variable, except for the fact that we replace x by x y and y Greek by y Greek, which gives us the following identity i i equals f g divided by x zero. We can therefore state the expression of the explicit transform Legendre g of f g is equal to f minus i i x y. We can also link the curves of two functions which are related by a Legendre transformation. These functions are the f-functions. And g. G. Is equal to f minus i x y. And therefore f is equal to.

Notes

Summary

4m 31s

Courbure

- **Fonctions d'état** $F\left(X_1,..,X_i,..,X_n\right)$ et $G\left(X_1,..,Y_i,..,X_n\right)$:

$$G\left(X_1,..,Y_i,..,X_n\right) = F\Big(X_1,..,X_i\left(X_1,..,Y_i,..,X_n\right),..,X_n\Big) - Y_i\,X_i\left(X_1,..,Y_i,..,X_n\right)$$

$$F\left(X_1,..,X_i,..,X_n\right) = G\Big(X_1,..,Y_i\left(X_1,..,X_i,..,X_n\right),..,X_n\Big) + X_i\,Y_i\left(X_1,..,X_i,..,X_n\right)$$

- **Dérivées partielles :**

$$\frac{\partial G}{\partial Y_i} = \frac{\partial F}{\partial X_i}\frac{\partial X_i}{\partial Y_i} - X_i - Y_i\frac{\partial X_i}{\partial Y_i} = -\,X_i$$

$$\frac{\partial F}{\partial X_i} = \frac{\partial G}{\partial Y_i}\frac{\partial Y_i}{\partial X_i} + Y_i + X_i\frac{\partial Y_i}{\partial X_i} = Y_i$$

- **Courbure (dérivées partielles secondes) :**

$$\frac{\partial^2 G}{\partial Y_i^2} = -\,\frac{\partial X_i}{\partial Y_i} \quad \text{et} \quad \frac{\partial^2 F}{\partial X_i^2} = \frac{\partial Y_i}{\partial X_i} \quad \Rightarrow \quad \boxed{\frac{\partial^2 G}{\partial Y_i^2} = -\left(\frac{\partial^2 F}{\partial X_i^2}\right)^{-1}}$$

Thermodynamique

I have more x y who. Given the dependence in terms of state variables of these two functions, we can now determine the partial derivatives of these functions. The partial derivative of g with respect to i i first is equal to the partial derivative of f with respect to x y times the derivative partial of x y with respect to i y minus here minus y i which multiplies the partial derivative of x with respect to i required, and then the partial derivative of f with respect to x. It is therefore the first term and the third term simplifies and we are left with the second term which is equal to minus x. Second, the partial derivative of f with respect to x y. It is the partial derivative of g with respect to i which sees the partial derivative of i with respect to x plus y i plus x y for the partial derivative of i Greek i with respect to x. And the partial derivative of g with respect to i Greek i, it is minus x which so the first term and the third term simplifies. We are left with the second term which is Greek i. We can now calculate the expression of the curvatures of these two functions with respect to the variable on which we perform the Legendre transformation.

Notes

Summary

6m 19s

Courbure

- **Fonctions d'état** $F(X_1,..,X_i,..,X_n)$ **et** $G(X_1,..,Y_i,..,X_n)$:

$$G(X_1,..,Y_i,..,X_n) = F\Big(X_1,..,X_i(X_1,..,Y_i,..,X_n),..,X_n\Big) - Y_i X_i(X_1,..,Y_i,..,X_n)$$
$$F(X_1,..,X_i,..,X_n) = G\Big(X_1,..,Y_i(X_1,..,X_i,..,X_n),..,X_n\Big) + X_i Y_i(X_1,..,X_i,..,X_n)$$

- **Dérivées partielles :**

$$\frac{\partial G}{\partial Y_i} = \frac{\partial F}{\partial X_i}\frac{\partial X_i}{\partial Y_i} - X_i - Y_i\frac{\partial X_i}{\partial Y_i} = -X_i$$
$$\frac{\partial F}{\partial X_i} = \frac{\partial G}{\partial Y_i}\frac{\partial Y_i}{\partial X_i} + Y_i + X_i\frac{\partial Y_i}{\partial X_i} = Y_i$$

- **Courbure (dérivées partielles secondes) :**

$$\frac{\partial^2 G}{\partial Y_i^2} = -\frac{\partial X_i}{\partial Y_i} \quad \text{et} \quad \frac{\partial^2 F}{\partial X_i^2} = \frac{\partial Y_i}{\partial X_i} \quad \Rightarrow \quad \boxed{\frac{\partial^2 G}{\partial Y_i^2} = -\left(\frac{\partial^2 F}{\partial X_i^2}\right)^{-1}}$$

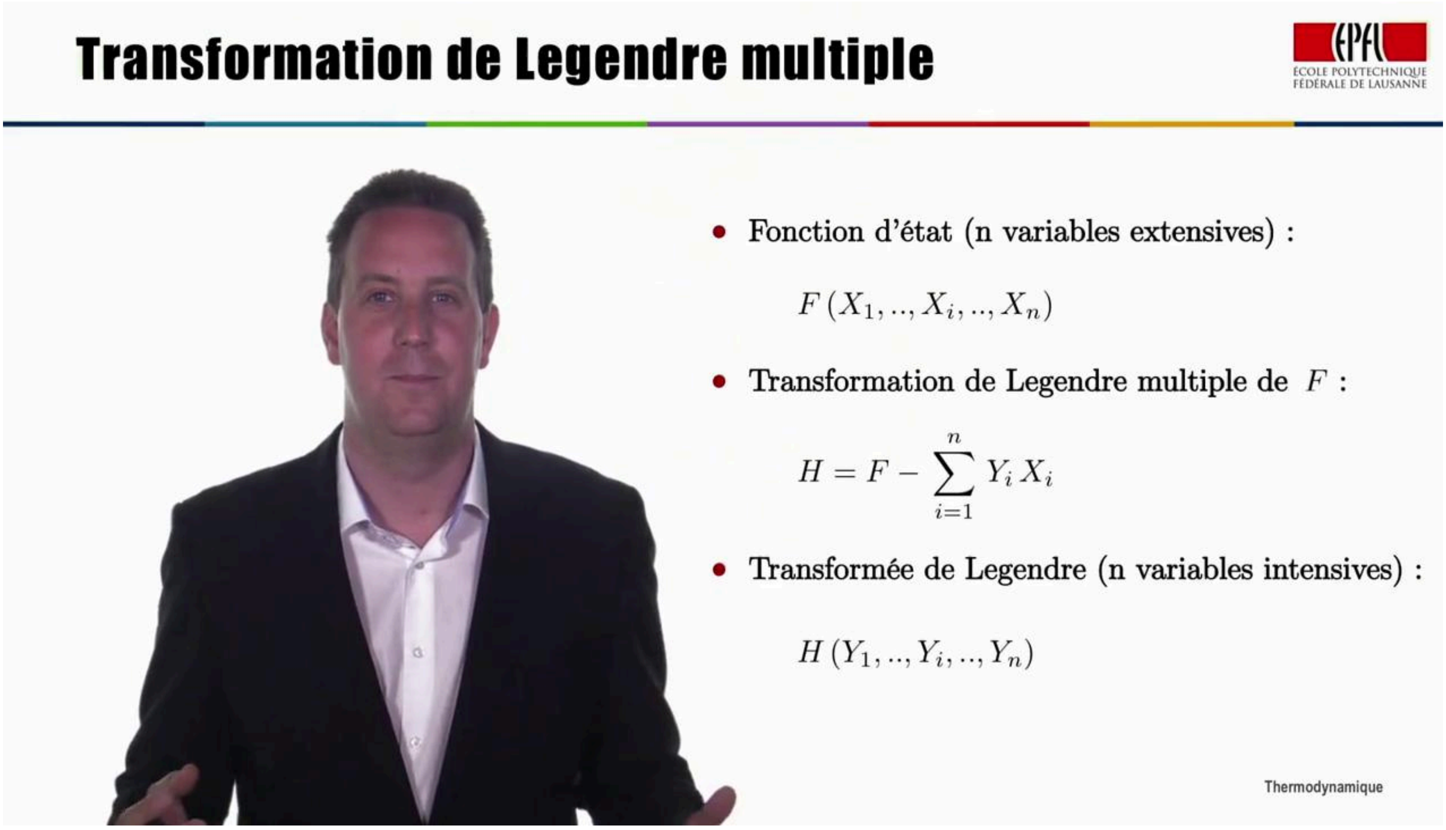
Thermodynamique

Curvature is the partial second derivative. Therefore the partial second derivative of g with respect to i Greek i is equal to the partial derivative of x with respect to i i. And then, in an analogous way, the second partial derivatives of f with respect to x y is equal to the partial derivative of i with respect to x. Therefore, we conclude that the partial second derivative of g with respect to i Greek y is equal to minus the inverse of the partial second derivative of f with respect to x. In other words, if we perform a Legendre transformation, the curvature of the Legendre transform has the opposite sign to the starting function and its norm is equal to the inverse of the norm of the starting function.

Notes

Summary

7m 46s



Finally. We will define a multiple Legendre transformation. We consider a state function of n extensive variables. This function is the function f which depends from x1 x2 to xn. And we perform a Legendre transformation. In the end, we obtain the function H which is equal to f months, the sum of y equal to n of y x and this function H. Which is the multiple Legendre transform depending on n intensive variables. H is therefore a function of Greek i, Greek i two etc. up to Greek y, n.

Notes

Summary

8m 37s